Partially-ordered multi-type algebras, multi-type frames and the category of polarities

Peter Jipsen, Chapman University

joint work with
F. Liang, Delft University of Technology, the Netherlands,
M. A. Moshier, Chapman University, California and
A. Tzimoulis, Delft University of Technology, the Netherlands

LATD: Logic, Algebra and Truth Degrees

University of Bern, Switzerland, August 31, 2018
Overview

- Partially-ordered multi-type algebras
- Semi-De Morgan algebras
- Multi-type frame semantics
- Term-equivalence for multi-type algebras
A series of papers about cut-free display calculi

Dynamic epistemic logic displayed [Greco, Kurz, Palmigiano, 2013]
Linear logic properly displayed [Greco and Palmigiano, 2016]
Lattice logic properly displayed [Greco and Palmigiano, 2016]
Bilattice logic properly displayed [Greco, Liang, Palmigiano and Rivieccio, 2017]
Multi-type display calculus for semi-DeMorgan logic [Greco, Liang, Moshier and Palmigiano, 2017]
Multi-type algebras

A **multi-type algebra** is of the form $\mathcal{A} = ((A_\tau)_{\tau \in \mathcal{T}}, \mathcal{F})$ where each $f \in \mathcal{F}$ is a function $f : A_{\tau_1} \times \cdots \times A_{\tau_n} \to A_\tau$ for some $\tau_1, \ldots, \tau_n, \tau \in \mathcal{T}$. 

The set of types $\mathcal{T}$ and the sequences $\tau_1, \ldots, \tau_n, \tau$ for each operation $f \in \mathcal{F}$ determine the signature $\Sigma$ of the algebra.

Multi-type algebras are also called many-sorted or heterogeneous algebras by Birkhoff and Lipson [1970]. They have applications, e.g., in algebraic logic and as abstract data types in computer science.
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Partially-ordered multi-type algebras

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A **partially-ordered** multi-type algebra (**pom-algebra** for short) is a multi-type algebra \( \mathbb{A} = ((A_\tau, \leq_\tau)_{\tau \in \mathcal{T}}, \mathcal{F}) \) where every \( (A_\tau, \leq_\tau) \) is a partially-ordered set and all operations are monotone.
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A **monotonicity type** for an $n$-ary operation $f$ is a sequence $\varepsilon_f \in \{1, \partial\}^{n+1}$ such that

$$A_{\varepsilon_f,i}^{\varepsilon_f,i} = \begin{cases} A_{\tau_i} & \text{if } \varepsilon_f,i = 1 \\ A_{\partial}^{\varepsilon_f,i} & \text{otherwise} \end{cases}$$

and $f : A_{\varepsilon_f,1}^{\varepsilon_f,1} \times \cdots \times A_{\varepsilon_f,n}^{\varepsilon_f,n} \rightarrow A_{\varepsilon_f,0}^{\varepsilon_f,0}$ is (pointwise) isotone.
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\end{cases}
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Pom-algebras are a generalization of (single-typed) po-algebras.

**Varieties** and **quasivarieties** of po-algebras were studied by Pigozzi [2004]
Let $f_{a,i}(x) = f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$ where $a_j \in A_{\tau_j}$ and $a = (a_1, \ldots, a_n)$. 
Lattice-ordered multi-type algebras

Let \( f_{a,i}(x) = f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) \) where \( a_j \in A_{\tau_j} \) and \( a = (a_1, \ldots, a_n) \).

A lattice-ordered multi-type algebra, or \( \ell m \)-algebra, is a multi-type algebra \( \mathbb{A} = ((A_{\tau}, \lor_{\tau}, \land_{\tau})_{\tau \in \mathcal{T}}, \mathcal{F}) \) where every \((A_{\tau}, \lor_{\tau}, \land_{\tau})\) is a lattice and every \( f : A_{\tau_1}^{\varepsilon_{f,1}} \times \cdots \times A_{\tau_n}^{\varepsilon_{f,n}} \rightarrow A_{\tau}^{\varepsilon_{f,0}} \) is \( \lor_{\varepsilon_{f,i}} \)-preserving in each argument \( i \).
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This means $f_{a,i}(x \lor^{\varepsilon_f,i} y) = f_{a,i}(x) \lor^{\varepsilon_f,0} f_{a,i}(y)$, where $\lor^{\partial} = \land$ and $\lor^1 = \lor$. 
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If constants $\bot, \top$ are part of the language, then also $f(\bot^{\varepsilon_{f,i}}) = \bot^{\varepsilon_{f,0}}$, where $\bot^d = \top$ and $\bot^1 = \bot$. 

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This means \( f_{a,i}(x \top_{\mathcal{F},i} y) = f_{a,i}(x) \top_{\mathcal{F},0} f_{a,i}(y) \), where \( \top_{\mathcal{F}} = \bot \) and \( \top_{\mathcal{F},1} = \top \).

If constants \( \bot, \top \) are part of the language, then also \( f(\bot_{\mathcal{F},i}) = \bot_{\mathcal{F},0} \), where \( \bot_{\mathcal{F}} = \top \) and \( \bot_{\mathcal{F},1} = \bot \).

These conditions are Sahlqvist, hence the variety of all \( \ell m \)-algebras is canonical.
Lattice-ordered multi-type algebras

Let \(f_{a,i}(x) = f(a_1, \ldots, a_i, x, a_{i+1}, \ldots, a_n)\) where \(a_j \in A_{\tau_j}\) and \(a = (a_1, \ldots, a_n)\).

A lattice-ordered multi-type algebra, or \(\ell m\)-algebra, is a multi-type algebra \(A = ((A_\tau, \vee_\tau, \wedge_\tau)_{\tau \in T}, \mathcal{F})\) where every \((A_\tau, \vee_\tau, \wedge_\tau)\) is a lattice and every \(f : A^{\epsilon_f,1}_{\tau_1} \times \cdots \times A^{\epsilon_f,n}_{\tau_n} \rightarrow A^{\epsilon_f,0}_\tau\) is \(\vee^{\epsilon_f,i}\)-preserving in each argument \(i\).

This means \(f_{a,i}(x \vee^{\epsilon_f,i} y) = f_{a,i}(x) \vee^{\epsilon_f,0} f_{a,i}(y)\), where \(\vee^\partial = \wedge\) and \(\vee^1 = \vee\).

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The decomposition of lattice-ordered unitype algebras into simpler loosely connected multi-type components can lead to decision procedures for the equational theory, using cut-free sequent calculi.
An example: Semi-De Morgan algebras

A semi-DeMorgan algebra \( A = (A, \lor, \land, \bot, \top, ') \) is a bounded distributive lattice \( (A, \lor, \land, \bot, \top) \) such that the following identities hold:

1. \((x \lor y)' = x' \land y'\)
2. \((x \land y)'' = x'' \land y''\)
3. \(x''' = x'\)
4. \(\bot' = \top\)
5. \(\top' = \bot\)

If the identity \((x'') = x\) holds then \( A \) is a DeMorgan algebra and 2.-5. are redundant.

Semi-DeMorgan algebras are from [Sankappanavar JSL 1987]
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Let $(S)DMA$ be the variety of (semi-)DeMorgan algebras.

DMA is easily seen to be canonical, i.e., closed under canonical extensions.
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DeMorgan image of a semi-DeMorgan algebra

For a semi-DeMorgan algebra $A$, let $B = \{x'' : x \in A\}$ and define $h : A \to B$ by $h(a) = a''$ and
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**Lemma 1**: $h(e(b)) = b$ and $e(h(a)) = a''$ for all $a \in A, b \in B$. 
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**Lemma 1:** \( h(e(b)) = b \) and \( e(h(a)) = a'' \) for all \( a \in A, b \in B \).

**Proof:** \( b \in B \) implies \( b = x'' \) for some \( x \in A \)
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Hence $h(e(b)) = h(b) = b'' = x''' = x'' = b$.
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$e(h(a)) = e(a'') = a''$ since $a'' \in B$. □
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Define $B = (B, \sqcup, \sqcap, 0, 1, -)$ by $x \sqcup y = h((e(x) \vee e(y))'')$
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For a semi-DeMorgan algebra $A$, let $B = \{x'' : x \in A\}$ and define $h : A \to B$ by $h(a) = a''$ and $e : B \to A$ by $e(b) = b$.

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Define $B = (B, \sqcup, \sqcap, 0, 1, \neg)$ by $x \sqcup y = h((e(x) \lor e(y))'')$

$x \sqcap y = h(e(x) \land e(y))$, $0 = h(\bot)$, $1 = h(\top)$, $x^- = h((e(x))')$.
DeMorgan image of a semi-DeMorgan algebra

For a semi-DeMorgan algebra $A$, let $B = \{ x'' : x \in A \}$ and define $h : A \rightarrow B$ by $h(a) = a''$ and $e : B \rightarrow A$ by $e(b) = b$.

**Lemma 1:** $h(e(b)) = b$ and $e(h(a)) = a''$ for all $a \in A, b \in B$.

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$x \sqcap y = h(e(x) \land e(y)),$ $0 = h(\bot),\quad 1 = h(\top), \quad x^{\neg} = h((e(x))')$.

**Lemma 2:** $B$ is a DeMorgan algebra.
DeMorgan image of a semi-DeMorgan algebra

For a semi-DeMorgan algebra $A$, let $B = \{x'' : x \in A\}$ and define $h : A \to B$ by $h(a) = a''$ and $e : B \to A$ by $e(b) = b$.

**Lemma 1:** $h(e(b)) = b$ and $e(h(a)) = a''$ for all $a \in A, b \in B$.

**Proof:** $b \in B$ implies $b = x''$ for some $x \in A$
Hence $h(e(b)) = h(b) = b'' = x'''' = x'' = b$.

Proof: $b'' = h(e(b)) = e(h(a)) = a''$ since $a'' \in B$.

Define $B = (B, \sqcup, \sqcap, 0, 1, -)$ by $x \sqcup y = h((e(x) \lor e(y))'')$
$x \sqcap y = h(e(x) \land e(y))$, $0 = h(\bot)$, $1 = h(\top)$, $x^- = h((e(x))')$.

**Lemma 2:** $B$ is a DeMorgan algebra.

**Proof:** $b^- = h((e(h((e(b))'))')) = h(e(b)'') = b''' = b'' = x'''' = x'' = b$. 
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Hence $h(e(b)) = h(b) = b'' = x''' = x'' = b$.
e(h(a)) = e(a'') = a''$ since $a'' \in B$.

Define $B = (B, \sqcup, \sqcap, 0, 1, \neg)$ by $x \sqcup y = h((e(x) \lor e(y)))''$
x $\sqcap y = h(e(x) \land e(y))$, \quad $0 = h(\bot)$, \quad $1 = h(\top)$, \quad $x = h((e(x))')$.

**Lemma 2:** $B$ is a DeMorgan algebra.

**Proof:** $b = h((e(h((e(b))'))')) = h(e(b)'') = b''' = b'' = x'''' = x'' = b$.
$(b \sqcup c)' = h((e(h((e(b) \lor e(c))))'))' = h((b \lor c)'') = (b \lor c)' = b' \land c'$
DeMorgan image of a semi-DeMorgan algebra

For a semi-DeMorgan algebra $A$, let $B = \{x'' : x \in A\}$ and define

$h : A \rightarrow B$ by $h(a) = a''$ and 

$e : B \rightarrow A$ by $e(b) = b$.

**Lemma 1:** $h(e(b)) = b$ and $e(h(a)) = a''$ for all $a \in A, b \in B$.

**Proof:** $b \in B$ implies $b = x''$ for some $x \in A$

Hence $h(e(b)) = h(b) = b'' = x''' = x'' = b$.

$e(h(a)) = e(a'') = a''$ since $a'' \in B$.

Define $B = (B, \sqcup, \sqcap, 0, 1, \neg)$ by 

$x \sqcup y = h((e(x) \lor e(y)))''$ 

$x \sqcap y = h(e(x) \land e(y)),$ 

$0 = h(\bot), 1 = h(\top), x = h((e(x))').$

**Lemma 2:** $B$ is a DeMorgan algebra.

**Proof:** $b'\neg = h((e(h((e(b')))'))) = h(e(b'')) = b''' = b'' = x''' = x'' = b$.

$(b \sqcup c)' = h((e(h((e(b) \lor e(c))))')) = h((e(b) \lor e(c))''' = (b \lor c)' = b' \land c' 

$b \neg \sqcap c = h(e(h((e(b]))) \land e(h((e(c)))))) = h((e(b))' \land h((e(c))') = 

b''' \land c''' = b' \land c'$. 

Peter Jipsen — Chapman University — LATD 2018 — August 31
A heterogeneous semi-DeMorgan algebra or HSM-algebra \((D, B, e, h)\) is a distributive lattice \(D\),
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Heterogeneous semi-DeMorgan algebras

A heterogeneous semi-DeMorgan algebra or HSM-algebra \((D, B, e, h)\) is a distributive lattice \(D\), a DeMorgan algebra \(B\), an injective map \(e : B \hookrightarrow D\) and a surjective lattice homomorphism \(h : D \twoheadrightarrow B\) such that

\[
e(b \sqcap c) = e(b) \land e(c),
\]

\[
e(0) = \perp,
\]

\[
e(1) = \top,
\]

\[
h(e(b)) = b.
\]

Lemma 3: If \(A\) is a semi-DeMorgan algebra and we define \(B\), \(e\), \(h\) as on the previous slide then \(F(A) = ((A, \lor, \land, 0, 1), B, e, h)\) is a HSM-algebra.

Proof:

\[
e(b \sqcap c) = b \sqcap c = h(e(b) \land e(c)) = h(b \land c) = (b \land c)\prime = e(b) \land e(c),
\]

\[
e(0) = e(h(\perp)) = \perp,
\]

\[
e(1) = e(h(\top)) = \top,
\]

\[
h(e(b)) = b\prime = x\prime = x = b.
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Lemma 4: If \(D = (D, B, e, h)\) is a HSM-algebra and \(x\prime = e(h(x))\) then \(G(D) = (D, \lor, \land, 0, 1, \prime)\) is a semi-DeMorgan algebra and \(FG = D, GF = A\).

With the standard definition of multi-type homomorphism, \(F\) and \(G\) are functors that give an equivalence of categories.
Heterogeneous semi-DeMorgan algebras

A heterogeneous semi-DeMorgan algebra or HSM-algebra \((D, B, e, h)\) is a distributive lattice \(D\), a DeMorgan algebra \(B\), an injective map \(e : B \rightarrow D\) and a surjective lattice homomorphism \(h : D \twoheadrightarrow B\) such that

\[
\begin{align*}
e(b \wedge c) &= e(b) \wedge e(c) \\
e(0) &= \perp \\
e(1) &= \top \\
h(e(b)) &= b
\end{align*}
\]

Lemma 3: If \(A\) is a semi-DeMorgan algebra and we define \(B, e, h\) as on the previous slide then \(F(A) = (\langle A, \lor, \land, 0, 1 \rangle, B, e, h)\) is a HSM-algebra.

Proof:

\[
e(b \wedge c) = b \wedge c = h(e(b) \land e(c)) = h(b \land c) = (b \land c)'' = e(b) \land e(c) = e(b \wedge c)
\]

\[
e(0) = e(h(\perp)) = \perp, \quad e(1) = e(h(\top)) = \top, \quad \text{and} \quad h(e(b)) = b'' = b
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Lemma 4: If \(D = (D, B, e, h)\) is a HSM-algebra and \(x'' = e((h(x))^{-})\) then \(G(D) = (D, \lor, \land, 0, 1, x'')\) is a semi-DeMorgan algebra and \(FG(D) \cong D\) and \(GF(A) \cong A\).

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Proof: \[e(b \cap c) = b \cap c = h(e(b) \land e(c)) = h(b \land c) = b \land c = e(b) \land e(c), \quad e(0) = \bot,\]

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Heterogenous semi-DeMorgan algebras are canonical

The ingredients of HSM-algebras are a distributive lattice $D$, a DeMorgan algebra $B$, an injective map $e : B \hookrightarrow D$ and a surjective lattice homomorphism $h : D \twoheadrightarrow B$ such that

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Each of these pieces is preserved by canonical extension.
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Hence semi-DeMorgan algebras are canonical.

Furthermore, [Greco, Liang, Moshier, Palmigiano 2017] provide a cut-free multi-type display calculus for HSM-algebras that has the sub-formula property.
A semi-DeMorgan lattice $L = (L, \lor, \land,')$ is a lattice $(L, \lor, \land,)$ with a unary operation $'$ such that:

1. $(x \lor y)' = x' \land y'$
2. $(x \land y)' = x' \land y'$
3. $x''' = x'$

If the identity $x'' = x$ holds then $L$ is a DeMorgan lattice and 2., 3. are redundant.
A semi-DeMorgan lattice $L = (L, \lor, \land, ')$ is a lattice $(L, \lor, \land, )$ with a unary operation $'$ such that

1. $(x \lor y)' = x' \land y'$

If the identity $x' = x$ holds then $L$ is a DeMorgan lattice and 2., 3. are redundant.
A **semi-DeMorgan lattice** \( L = (L, \vee, \wedge, ') \) is a lattice \( (L, \vee, \wedge, ) \) with a unary operation ‘ such that

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2. \( (x \land y)'' = x'' \land y'' \)

If the identity \( x'' = x' \) holds then \( L \) is a **DeMorgan lattice** and 2. is redundant.

Let \( (S)DML \) be the variety of (semi-)DeMorgan lattices. DML is easily seen to be canonical, i.e., closed under canonical extensions.
Semi-De Morgan lattices

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A **semi-DeMorgan lattice** decomposes as a lattice and a DeMorgan lattice, connected by maps $e$, $h$ as before.

A **heterogeneous semi-DeMorgan lattice** or HSM-lattice $(L, M, e, h)$ is a lattice $L$, a DeMorgan lattice $M$, an injective map $e : M \rightarrow L$ and a surjective lattice homomorphism $h : L \rightarrow M$ such that $e(x \lor y) = e(x) \land e(y)$, and $h(e(x)) = x$ for all $x, y \in M$.

Again a categorical equivalence can be established between semi-DeMorgan lattices and HSM-lattices.

For another example, a linear logic algebra with exponentials decomposes into a commutative residuated lattice and a Heyting algebra connected by appropriate maps, which leads to a cut-free display calculus for linear logic [Greco and Palmigiano 2016].
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Residuated lattices and frames

Residuated frames were defined in [Galatos and J., 2013] for unisorted residuated lattices.
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A residuated lattice is an algebra \( A = (A, \lor, \land, \cdot, \backslash, /, 1) \) such that \((A, \lor, \land)\) is a lattice, \((A, \cdot, 1)\) is a monoid and for all \(x, y, z \in A\) \(xy \leq z \iff y \leq x \backslash z \iff x \leq z / y\).

A residuated frame is a 2-sorted ternary relational structure \( W = (W, W', N, \circ, \backslash, /, E) \) such that \((W, W', N)\) is a polarity, \(\circ \subseteq W^3 \subseteq W \times W' \times W\), \(\backslash \subseteq W' \times W^2\) and for all \(x, y, z \in W\) and \(w \in W'\)

\[ N \uparrow (E \circ x) = N \uparrow \{x\} = N \uparrow (x \circ E) \]

where \(N \uparrow X = \{y \in W' : \forall x \in X, xNy\}\).

Note \(x \circ y = \{z : \circ (x, y, z)\}\), \(X \circ Y = \{z : \circ (x, y, z), x \in X, y \in Y\}\).

3. implies \(x \cdot y = \gamma N(x \circ y)\) is residuated on \(W^+ = (\gamma N[P(W)], \lor, \cap, \cdot, \backslash, /, 1)\) where \(\gamma N(X) = N \downarrow N \uparrow (X)\), \(X \setminus Z = \{y \in W : X \circ \{y\} \subseteq Z\}\), \(1 = \gamma N(E)\).
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1. \( N^\uparrow(E \circ x) = N^\uparrow\{x\} = N^\uparrow(x \circ E) \) where \( N^\uparrow(X) = \{ y \in W' : \forall x \in X, xNy \} \)
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1. $N^\uparrow(E \circ x) = N^\uparrow\{x\} = N^\uparrow(x \circ E)$ where $N^\uparrow(X) = \{y \in W' : \forall x \in X, xNy\}$
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Residuated lattices and frames

Residuated frames were defined in [Galatos and J., 2013] for unisorted residuated lattices.

A residuated lattice is an algebra $A = (A, \lor, \land, \cdot, \backslash, /, 1)$ such that $(A, \lor, \land)$ is a lattice, $(A, \cdot, 1)$ is a monoid and for all $x, y, z \in A$  $x y \leq z \iff y \leq x \backslash z \iff x \leq z / y$.

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Note $x \circ y = \{z : \circ(x, y, z)\}$, $X \circ Y = \{z : \circ(x, y, z), x \in X, y \in Y\}$
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3. implies $x \cdot y = \gamma_N(x \circ y)$ is residuated on $W^+ = (\gamma_N(\mathcal{P}(W)), \lor, \cap, \cdot, \backslash, /, 1)$

where $\gamma_N(X) = N^\downarrow N^\uparrow(X)$, $X \backslash Z = \{y \in W : X \circ \{y\} \subseteq Z\}$, $1 = \gamma_N(E)$. 
Algebraic semantics for lattice-ordered multi-sorted algebras

**Residuated frames** extend modal Kripke frames to nondistributive logics and certain lattice-ordered algebras.
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From the HSM-lattice \((L, M, e, h)\) it is clear that we need a lattice polarity for \(L\) and a DeMorgan polarity for \(M\).
A **heterogenous semi-DeMorgan frame** is of the form 
$((V, V', P), (W, W', N, ??), ??, ??)$

How do we represent $V$, $V'$, $P$, $W$, $W'$, $N$, $??$ in the frame?
Fortunately Drew Moshier has provided the appropriate notion of morphism for polarities.
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Frame morphisms

Complete lattices with complete homomorphisms form a category
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What are the *appropriate* morphisms for polarity frames?
Frame morphisms

Complete lattices with complete homomorphisms form a category

What are the appropriate morphisms for polarity frames?

For a frame $\mathcal{W} = (\mathcal{W}, \mathcal{W}', \mathcal{N})$ the relation $\mathcal{N}$ is an identity morphism that induces the identity map $\gamma = \mathcal{N}_\downarrow \mathcal{N}_\uparrow$ on the closed sets
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For a frame $\mathbf{W} = (W, W', N)$ the relation $N$ is an identity morphism that induces the identity map $\gamma = N\downarrow N\uparrow$ on the closed sets

A frame morphism $R : \mathbf{V} \rightarrow \mathbf{W} = (V, W', R)$ is a relation $R \subseteq V \times W'$ such that $N\downarrow_v N\uparrow_v R\downarrow = R\downarrow = R\downarrow N\uparrow_w N\downarrow$
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or equivalently $R_\uparrow N_\downarrow V N_\uparrow V = R_\uparrow = N_\uparrow W N_\downarrow W R_\uparrow$

In either case we say that $R$ is compatible.
Compatible morphisms ≡ meet-semilattice homomorphisms

Lemma. If $R$ is compatible then $R \downarrow N_{\uparrow} : W^+ \to V^+$ preserves $\bigcap$.
Compatible morphisms $\equiv^\partial$ meet-semilattice homomorphisms

**Lemma.** If $R$ is compatible then $R \downarrow N_W^\uparrow : W^+ \to V^+$ preserves $\bigcap$

$$
\begin{array}{c}
\begin{tikzpicture}
  \node (V) at (0,0) {$V$};
  \node (W) at (2,0) {$W$};
  \node (NW) at (1.1,1) {$N_W$};
  \node (NW') at (1.1,2) {$N_W$};
  \draw[->,dotted] (NW) -- (NW');
  \draw[->] (NW) -- (V);
  \draw[->] (NW) -- (W);
  \draw[->] (NW') -- (V);
  \draw[->] (NW') -- (W);
  \node at (1,1.5) {$R$};
\end{tikzpicture}
\end{array}
$$

**Lemma.** If $R$ is compatible then $N_W^\downarrow R^\uparrow : V^+ \to W^+$ preserves $\bigvee$

$$
\begin{array}{c}
\begin{tikzpicture}
  \node (V) at (0,0) {$V$};
  \node (W) at (2,0) {$W$};
  \node (V+) at (0,2) {$V^+$};
  \node (W+) at (2,2) {$W^+$};
  \draw[->] (V) -- (V+);
  \draw[->] (W) -- (W+);
  \draw[->] (V+) -- (V); \\
  \node at (1,1.5) {$\gamma$};
\end{tikzpicture}
\end{array}
$$

$V_\_ = \gamma[U_\_]$
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\[ N_V \quad R \quad N_W \]
\[ V \quad W \quad V^+ \quad W^+ \]

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Conversely, given a completely join-preserving map $h : V^+ \to W^+$ define $xRy \iff y \in h(\gamma_N\{x\})$. Then $R$ is compatible.
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\[
\begin{array}{c}
\text{V} \\
\text{N}_W
\end{array}
\quad
\begin{array}{c}
\text{W} \\
\text{R}
\end{array}
\quad
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\text{N}_W
\end{array}
\quad
\begin{array}{c}
\text{W^+} \\
\text{V^+}
\end{array}
\]

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\[
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$\implies$ The category of frames with compatible morphisms is **equivalent** to the category of complete lattices with completely join-preserving maps.
Lattice compatible morphisms

Lemma: $R \downarrow N_W \uparrow\downarrow \text{ preserves } \lor \text{ iff there exists a compatible relation } R_* : W \to V' \text{ such that } R \downarrow N_W \uparrow = N_V \downarrow R_* \uparrow \text{ (call } R \text{ lattice compatible)}$
Lattice compatible morphisms

**Lemma:** $R \downarrow N_{W} \uparrow$ preserves $\lor$ iff there exists a compatible relation $R_{*}: W \rightarrow V'$ such that $R \downarrow N_{W} \uparrow = N_{V} \uparrow R_{*}$ (call $R$ lattice compatible)

**Theorem [Moshier]:** The category $\text{LPFrm}$ of all frames with lattice compatible relations as morphisms is dual to the category $\text{CLat}$ of complete lattices with complete lattice morphisms.
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**Lemma:** $R \downarrow N_{W}^{\uparrow}$ preserves $\vee$ iff there exists a compatible relation $R_{\ast} : W \to V'$ such that $R \downarrow N_{W}^{\uparrow} = N_{V}^{\downarrow} R_{\ast}^{\uparrow}$ (call $R$ lattice compatible)

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Note that every morphism has itself as epi-mono factorization

\[
\begin{array}{ccc}
V' & R \cong & W' \\
\downarrow & & \downarrow \\
N_{V} & = & N_{W} \\
\uparrow & & \uparrow \\
V & & W
\end{array}
\]
A **heterogenous semi-DeMorgan frame** is of the form

\((V, V', N_V), (W, W', N_W, Q, Q_*), F, H, H_*\) where

Application: semi-DeMorgan lattices are closed under MacNeille completions and canonical extensions. Use this framework to design a cut-free Gentzen system with the subformula property and check if this gives a decision procedure for equations of semi-DeMorgan lattices.
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How are uni-type and multi-type algebras related?

Syntactic methods are signature/presentation dependent
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A *multi-sorted Lawvere theory* for a set of types \( T \) is a small category \( C \) with finite products such that the objects are all finite products of the types in \( T \). A *multi-type algebra* for \( C \) is a product preserving functor \( A : C \rightarrow \text{Set} \) (the category of sets). For any morphism \( f \), the function \( A(f) \) is an operation of the algebra.
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Peter Jipsen — Chapman University — LATD 2018 — August 31
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A multi-type algebra for $\mathcal{C}$ is a product preserving functor $A : \mathcal{C} \rightarrow \text{Set}$ (the category of sets). For any morphism $f$, the function $Af$ is an operation of the algebra.
Lawvere theories for po-multi-type algebras

A pom-algebra for $\mathcal{C}$ is a product preserving functor $A : \mathcal{C}_\partial \to \text{Pos}_\partial$ (the category of posets with order-preserving maps expanded with a natural transformation $\partial_P : P \to P^\partial$ given by $\partial_P(x) = x$, and similarly for $\mathcal{C}_\partial$).
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A **$\ell m$-algebra** is a pom-algebra that satisfies the join-preservation identities.

In this setting, term-equivalence for multi-sorted algebras is given by categorical equivalence of their Lawvere theories.
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Lawvere theories for po-multi-type algebras

A pom-algebra for $C$ is a product preserving functor $A : C_\partial \to \text{Pos}_\partial$ (the category of posets with order-preserving maps expanded with a natural transformation $\partial_P : P \to P^\partial$ given by $\partial_P(x) = x$, and similarly for $C_\partial$).

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This is another tool to adjust the presentation of a logic (or class of algebras) to possibly make it easier to work with.
Some references


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https://orion.math.iastate.edu/dpigozzi/notes/santiago_notes.pdf

Thanks!

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